International Journal of Theoretical Physics, Vol. 40, No. 7, 2001

Unified Treatment of Spacelike and Timelike *SO*(3, 1) Yang-Mills Fields

A. Reşit Dündarer[1]

Received October 18, 2000

SO(3, 1) valued Yang-mills fields stemming from spacelike and timelike vectors that were studied separately in earlier works are unified by introducing a parameter λ that takes values in the interval $[-1, 1]$.

In this paper, the early works in which *SO*(3, 1) valued Yang-Mills fields were constructed using the Dirac algebra valued field $w = w^\mu \gamma_\mu$ with either w a timelike vector ($w^2 = (w^0)^2 - \vec{w} \cdot \vec{w} < 0$) (Dündarer, 1993) or a spacelike vector ($w^2 > 0$) (Dündarer, 2001) are unified by introducing a parameter $\lambda = \pm 1$, and then generalizing λ to take a value in the domain $[-1, 1]$, the endpoints of which give the previous timelike or spacelike fields.

To review, we have (Dündarer, 1993, 2001) the gauge potential A and the corresponding Yang-Mills field $F = dA + A \wedge A$ as

$$\text{timelike:} \qquad w^2 < 0: \quad A = -\frac{[w, dw]}{2(1 - w^2)} \Rightarrow F = -\frac{dw \wedge dw}{(1 - w^2)^2}, \tag{1}$$

$$\text{spacelike:} \qquad w^2 > 0: \quad A = +\frac{[w, dw]}{2(1 + w^2)} \Rightarrow F = +\frac{dw \wedge dw}{(1 + w^2)^2} \tag{2}$$

which can compactly be written as

$$A_\pm(w) \equiv \frac{\pm[w, dw]}{2(1 \pm w^2)} \Rightarrow F_\pm(w) = \frac{\pm dw \wedge dw}{(1 \pm w^2)^2}. \tag{3}$$

Now we introduce a fixed parameter $\lambda \in [-1, 1]$ and define

$$A_\lambda(w) \equiv \frac{\lambda[w, dw]}{2(1 + \lambda w^2)} \tag{4}$$

[1] Physics Department, Middle East Technical University, Ankara 06531, Turkey.

0020-7748/01/0700-1359$19.50/0 © 2001 Plenum Publishing Corporation

from which one obtains the Yang-Mills field F as

$$F_\lambda(w) = \frac{\lambda dw \wedge dw}{(1+\lambda w^2)^2}. \tag{5}$$

As to the vector components w^μ, we parametrize them in terms of $(\sigma, \tau) \in M_2$ and $z \in CP_1$ as

$$w^0 = \frac{\sigma}{2}(e^\tau + \lambda e^{-\tau}), \tag{6}$$

$$w^k = \frac{\sigma}{2}(e^\tau - \lambda e^{-\tau})u^k, \tag{7}$$

where

$$u^1 = \frac{\overline{z}^n + z^n}{1+(z\overline{z})^n}, \tag{8}$$

$$u^2 = \frac{i(\overline{z}^n - z^n)}{1+(z\overline{z})^n}, \tag{9}$$

$$u^3 = \frac{1-(z\overline{z})^n}{1+(z\overline{z})^n} \tag{10}$$

are stereographic coordinates of $S_2 \sim CP_1$ with $\vec{u}\cdot\vec{u} = 1$. This parametrization is consistent with previous works, as for $\lambda = -1$ we obtain $w^\mu = w^\mu_-$ (Dündarer, 1993) and for $\lambda = +1$ we obtain $w^\mu = w^\mu_+$ (Dündarer, 2001). Since

$$w^2 = (w^0)^2 - \vec{w}\cdot\vec{w} = \lambda\sigma^2 \tag{11}$$

the case $\lambda > 0$ corresponds to spacelike vector ($w^2 > 0$), $\lambda < 0$ corresponds to timelike vector ($w^2 < 0$), and $\lambda = 0$ corresponds to lightlike vector ($w^2 = 0$). Note that $\lambda = 0$ gives null fields $A = F = 0$.

As to the range of parameters we set $\sigma \in [0, \Sigma]$ and $\tau \in [0, \Lambda]$ the upper bounds being a necessity of the finiteness of the universe. We also have $z = \rho\, e^{i\theta}$, with $\rho \in [0, \infty]$ and $\theta \in [0, 2\pi]$ as before.

We find the λ-dependent Yang-Mills field

$$F_\lambda(w) = \frac{-i\lambda\sigma_{\mu\nu}}{(1+\lambda^2\sigma^2)^2}\, dw^\mu \wedge dw^\nu \tag{12}$$

as

$$\begin{aligned} F_\lambda(w) = {} & \frac{-2\lambda^2\sigma}{(1+\lambda^2\sigma^2)^2} b'\, d\sigma \wedge d\tau \\ & + \frac{2\lambda\sigma^2}{(1+\lambda^2\sigma^2)^2} \cdot \frac{n^2(z\overline{z})^{n-1}(e^{2\tau} + \lambda^2\, e^{-2\tau} - 2\lambda)}{[1+(z\overline{z})^n]^2} b\, dz \wedge d\overline{z} \\ & + \frac{2n\lambda\sigma(e^\tau - \lambda e^{-\tau})}{(1+\lambda^2\sigma^2)^2[1+(z\overline{z})^n]}[(e^\tau f_{\mathrm{R}} - \lambda\, e^{-\tau} f_{\mathrm{L}})\overline{z}^{n-1}\, d\sigma \wedge d\overline{z} \end{aligned}$$

$$+\sigma(e^{\tau} f_{\mathrm{R}} + \lambda\, e^{-\tau} f_{\mathrm{L}})\bar{z}^{n-1}\, d\tau \wedge d\bar{z} - \left(e^{\tau} f_{\mathrm{L}}^{\dagger} - \lambda\, e^{-\tau} f_{\mathrm{R}}^{\dagger}\right) z^{n-1}\, d\sigma \wedge dz$$
$$-\sigma\left(e^{\tau} f_{\mathrm{L}}^{\dagger} + \lambda\, e^{-\tau} f_{\mathrm{R}}^{\dagger}\right) z^{n-1}\, d\tau \wedge dz] \tag{13}$$

where we defined the right- and left-handed fields $f_{\mathrm{R,L}}$ as

$$f_{\mathrm{R}}^{\dagger} = \frac{1+\gamma_5}{2} f^{\dagger} = \frac{1}{2}(f^{\dagger} + f'^{\dagger}), \qquad f_{\mathrm{R}} = \frac{1}{2}(f + f'), \tag{14}$$

$$f_{\mathrm{L}}^{\dagger} = \frac{1-\gamma_5}{2} f^{\dagger} = \frac{1}{2}(f^{\dagger} - f'^{\dagger}), \qquad f_{\mathrm{L}} = \frac{1}{2}(f - f') \tag{15}$$

with f and f' as defined in (Dündarer, 2001).

As a result of the duality relation satisfied by $\sigma_{\mu\nu}$ this field also satisfies

$$^{*}F = -i\gamma_5 F. \tag{16}$$

Then the Yang-Mills action

$$I_{\text{Y-M}} = \frac{1}{2}\int \mathrm{Tr}\,(F \wedge^{*} F) \tag{17}$$

becomes

$$I_{\text{Y-M}} = -2 \cdot 4! \int \frac{\lambda^2}{(1+\lambda^2\sigma^2)^4}\, d^4 w \tag{18}$$

$$= -(4\pi n)(4!) \int_0^{\infty} \frac{n\rho^{2n-1}\, d\rho}{(1+\rho^{2n})^2} \int_0^{\Sigma} \frac{\lambda^3\sigma^3\, d\sigma}{(1+\lambda^2\sigma^2)^4}$$
$$\times \int_0^{\Lambda} (e^{2\tau} + \lambda^2 e^{-2\tau} - 2\lambda)\, d\tau. \tag{19}$$

Performing the integrals we obtain

$$I_{\text{Y-M}} = -\frac{2\pi n}{\lambda}\left[1 - \frac{1+3\lambda^2\Sigma^2}{(1+\lambda^2\Sigma^2)^3}\right][\lambda^2 - 1 - 4\lambda\Lambda + e^{2\Lambda} - \lambda^2\, e^{-2\Lambda}]. \tag{20}$$

For $\lambda = \pm 1$ this expression reduces to

$$I_{\text{Y-M}}^{\pm} = 4\pi n\left[1 - \frac{1+3\Sigma^2}{(1+\Sigma^2)^3}\right][2\Lambda \mp \sinh 2\Lambda] \tag{21}$$

and in the limit $\lambda \to 0$ the lowest order terms are

$$I_{\text{Y-M}} \to -6\pi n\lambda^3\Sigma^4(e^{2\Lambda} - 1) \tag{22}$$

which tends to zero as required.

Finally at the conclusion I want to remark that in this paper λ was taken to be a fixed parameter and the case where it is also subject to variation is an open problem. That is, can one reach from one zone to the other (e.g., from spacelike field to the timelike) by continuously varying λ (e.g., from $+1$ to -1 in this case)?

If this turns out to be possible then it means that a tachionic field can turn into a spacelike one and vice versa.

REFERENCES

Reşit Dündarer, A. (1993). Finite action solutions on a four dimensional manifold with Minkowskian metric. *Doğa—Tr. Journal of Physics* **17** 97–102.

Reşit Dündarer, A. (2001). *SO*(3, 1) valued Yang-Mills fields. *International Journal of Theoretical Physics*. **40**, 507–516.